Adiabatic Rotation, Quantum Search and Preparation of Superposition States*

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We introduce the idea of using adiabatic rotation to generate superpositions of a large class of quantum states. For quantum computing this is an interesting alternative to the well-studied "straight line" adiabatic evolution. In ways that complement recent results, we show how to efficiently prepare three types of states: Kitaev's toric code state, the cluster state of the measurement-based computation model and the history state used in the adiabatic simulation of quantum circuit. We also show that the method, when adapted for quantum search, provides quadratic speedup as other optimal methods do with the advantages that the problem Hamiltonian is time-independent and that the energy gap above the ground state is strictly nondecreasing with time. Likewise the method can be used for optimization as an alternative to the algorithm of Farhi et al [1].

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I. INTRODUCTION

The presence of superpositions of classical states lies at the heart of many non-classical behaviors of quantum systems. It is therefore not surprising that uniform superposition states serve as powerful resources for quantum information processing. The main result in this article is that there is a conceptually simple way of preparing a class of linear superpositions by adiabatically rotating a driving Hamiltonian. Seen another way, it is also a method for finding the ground states of certain Hamiltonians. While the applications we discuss are all related to quantum computing, the idea is relevant to preparation of quantum states in general.

Adiabatic evolution studied in quantum computing often takes the form of a linear interpolation $H(s) = (1-s)H_{initial} + sH_{final}$ whether it is for problem solving [1, 2] or state preparation [3, 4, 5]. In this article we consider a different paradigm that is inspired by observations in [6] and follows a "rotation" instead. A time-dependent similarity transform on the entire Hamiltonian, as pointed out in [6], often requires highly nonlocal interactions and thus seems to have few applications. One may however adopt a different perspective and consider this time-dependent transformation as a driving term in the presence of a well-gapped and time-independent Hamiltonian. What we will show in Section II is that some of the desirable properties associated with the similarity transform remains in this more general setting.

In Section III, we outline how to use this idea to prepare states with interesting physical properties. The first example is the ground states of Kitaev's toric code [7], which have been of much interest because of their natural fault tolerance as memory and exhibition of topological order[8]. In [3] it is shown that a ground state of the toric code Hamiltonian can be prepared adiabatically through an linear interpolation in time $O(\sqrt{n})$ where n is the number of sites. Both this method and the original preparation-by-measurement method [9] are optimal since they saturate the Lieb-Robinson bound[10], which places a theoretical limit on the efficiency of any preparation method for topologically ordered states. However it requires duality mapping to an Ising model, which means the same result cannot be easily reproduced for a different system with topological order. We will show, without requiring extra types of interactions, that there is an elementary, circuit-like adiabatic evolution path that also prepares the toric code state optimally. The same idea would not only be applicable to other topologically ordered states, but also, for instance the cluster state [13] used in measurement-based computing. Recently it was shown in [4] that the a state similar to the cluster state in computation power can be prepared efficiently via adiabatic linear interpolation. Our method shows how the cluster state itself can be prepared adiabatically and efficiently.

In Section IV, we move on to a more complicated example. In [6] it is noted that the adiabatic linear interpolation corresponding to a quantum circuit used by [5] can be thought of as a special case in a larger family of paths. It may still appear, however, that fairly elaborate gap analysis [5, 11] would be required to check the efficiency of any path that generates the history state. This turns out to be not the case. We will describe two paths that generate the history state efficiently, and apart from being amenable to gap analysis, one only requires time-dependent interactions on a few qubits.

In Section V, we turn our attention to problem solving. Adiabatic quantum computing was originally proposed not for state preparation but to solve optimization (especially NP-complete) problems. While there is still controversy on the value of this approach, it is well-known

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that the the closely related adiabatic quantum search method [2] yields an optimal result, in the sense that it is as efficient as Grover's algorithm. Here we present an adiabatic rotation version of quantum search. Like the linear interpolation version, it is also optimal. The difference is that in our case, the spectral gap between the ground state and the first excited state is strictly nondecreasing, and this feature can be extended to other optimization problems. Clearly this would make it easier for us to determine whether a given problem can be solved efficiently. Our approach, however, has an additional difficulty compared to linear interpolation. While exact examples of linear interpolation algorithms often assume nonlocal interactions (as [2] does), one can almost always formulate a linear interpolation algorithm with local interactions. The situation is different with adiabatic rotation, because it requires separation of the initial state from the computational states. We will conclude with a brief discussion of what this means and some suggestions for future direction.

II. THE BASIC FORMULATION

In the simplest case, our Hamiltonian consists of a time-independent piece and a driving term under time-dependent similarity transform:

$$H(\theta) = H_0 + U_m(\theta) h_m U(\theta)_m^\dagger \tag{1}$$

where θ is a time-dependent rotation parameter and H_0 is a Hamiltonian with two degenerate ground states of zero energy. Suppose one ground state is $|\psi_a\rangle = \sum_{i=1}^n a_i |\gamma_i\rangle$ and another is $|\psi_b\rangle = |\gamma_m\rangle$ where $\{|\gamma_1\rangle ... |\gamma_n\rangle, |\gamma_m\rangle\}$ spans the effective Hilbert space of interest. The θ-dependent term acts on a two-dimensional space spanned by $\{|\gamma_m\rangle, |\gamma_j\rangle\}$ for some $1 \le j \le n$. That is, we define $h_m = K |\gamma_m\rangle \langle\gamma_m|$ for some positive constant K and $U(\theta)$ is a rotation in $\{|\gamma_m\rangle, |\gamma_j\rangle\}$:

$$U_m(\theta) = e^{i\mathbf{J}\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

$$U_m h_m U_m^\dagger = K \begin{pmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \tag{2}$$

Here $\mathbf{J}$ denotes the generator of rotation. One can check that

$$|0\rangle = \frac{1}{\sqrt{1 + a_j^2 \tan^2\theta}} (a_j \tan\theta |\gamma_m\rangle + \sum_{i=1}^n a_i |\gamma_i\rangle), \tag{3}$$

is a ground state of $H(\theta)$ with zero energy for any θ. We will consider θ between 0 and $\frac{\pi}{2}$.

In adiabatic evolution, the transition rate from a ground state $|0\rangle$ to an excited state $|k\rangle$ is well-known to be bounded by $\left|\langle k| \frac{dH}{dt} |0\rangle / (E_k - E_0)^2\right|^2$[12]. To keep this quantity constant as we vary θ, we write $\frac{dH}{dt} = \frac{dH}{d\theta}\frac{d\theta}{dt}$ and it follows that we need

$$\frac{d\theta}{dt} \sim \frac{(E_k(\theta) - E_0(\theta))^2}{\left|\langle k| \frac{dH}{d\theta} |0\rangle\right|}. \tag{4}$$

Our model offers at least two advantages if we wish to adiabatically manipulate a state of the form (3).

Claim 1 *Let* $g_k(\theta) = |E_k(\theta) - E_0(\theta)|$. *For* $0 < \theta < \frac{\pi}{2}$, $g_k'(\theta) \neq 0$.

Proof: If $|k(\theta)\rangle$ is an eigenstate of $H(\theta)$ with energy $E_k(\theta)$, by first-order perturbation theory,

$$\frac{dE_k(\theta)}{d\theta} \sim \langle k(\theta)| \frac{dH(\theta)}{d\theta} |k(\theta)\rangle$$

where $\frac{dH(\theta)}{d\theta}$ can be explicitly evaluated as

$$\begin{aligned} \frac{dH(\theta)}{d\theta} &= e^{i\mathbf{J}\theta}[i\mathbf{J}, H]e^{-i\mathbf{J}\theta} \\ &= \begin{pmatrix} -2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \\ \sin^2\theta - \cos^2\theta & 2\cos\theta\sin\theta \end{pmatrix} \end{aligned} \tag{5}$$

One readily checks that for $0 < \theta < \frac{\pi}{2}$ the expectation value of this operator vanishes if and only if the $|\gamma_m\rangle$ and $|\gamma_j\rangle$ components of the state satisfies the $\tan\theta$ ratio. If an eigenstate of $H(\theta)$ satisfies this, it would have to be the ground state which stays at zero. This means the gap only shifts in one direction. ■

Claim 2 $\left|\langle k| \frac{dH}{d\theta} |0\rangle\right| \le O(\sqrt{g_k(\theta)})$.

Proof: The idea of the proof is that $\frac{dH}{d\theta}|0\rangle$ couples to a state with high energy, and this limits its coupling into low lying states during the evolution. Using (5), we obtain

$$\begin{aligned} \frac{dH}{d\theta}|0\rangle &= \frac{a_j}{\sqrt{\cos^2\theta + a_j^2\sin^2\theta}} |p\rangle, \\ |p\rangle &\equiv \sin\theta |\gamma_j\rangle - \cos\theta |\gamma_m\rangle. \end{aligned}$$

The energy of an excited state is $g_k(\theta) = |\langle p|k\rangle|^2 K + \langle k| H_0 |k\rangle$, from which we see that $|\langle p|k\rangle| \le \sqrt{g_k(\theta)/K}$. ■

The first fact enables us to estimate efficiency by looking only at the end points. The second allows a higher speed for $d\theta/dt$ when the gap is small. A more detailed estimate yields:

$$\begin{aligned} &\left|\langle k| \frac{dH}{d\theta} |0\rangle\right|^2 \le A[(1+r^2)c_{mk}^2 g_1 - g_1 + g_k(\theta)] \\ &A \equiv \frac{a_j^2}{K(\cos^2\theta + a_j^2\sin^2\theta)} \\ &r \equiv \frac{\sum_{i=1}^n a_i \langle\gamma_i|0\rangle}{\langle\gamma_m|0\rangle}, \quad c_{mk} \equiv \langle\gamma_m|k\rangle \end{aligned} \tag{6}$$

where g_1 is the first excited state energy of H_0. We see that the quantities A and c_{mk} can further suppress the transition rate as they may contain factors inversely proportional to the size of the system.

To see why this is useful, consider the case $a_i = \frac{1}{\sqrt{n}}$ for $i = 1..n$ and θ running from 0 to $\frac{\pi}{4}$ adiabatically. It is clear that we can turn the state $\frac{1}{\sqrt{n}}(|\gamma_1\rangle + ... + |\gamma_n\rangle)$ to $\frac{1}{\sqrt{n+1}}(|\gamma_1\rangle + ... + |\gamma_n\rangle + |\gamma_m\rangle)$, thus adding an element to the uniform superposition. Reverse the process and we will delete the element from the superposition. Clearly, repeated applications of such processes with different driving terms allow us to prepare a large class of states. Generally speaking, if we wish to create a particular superposition of certain basis states, we can look at the Hamiltonian interactions we have control over that act either diagonally or off-diagonally on the basis. This set of interactions can then be thought of as a generating set for group elements that we want to eventually appear in the superposition. In the next section we will consider some examples to illustrate this idea.

II. THE TORIC CODE AND THE CLUSTER STATE

The ground state of the toric code Hamiltonian[7] can be generated efficiently by adiabatic linear interpolation [3]. The minimal gap in the evolution is estimated to be of order $O(n^{-\frac{1}{2}})$. Here we show a different path where the gap remains constant but the evolution time is also $O(\sqrt{n})$, thus also saturating the Lieb-Robinson bound [10]. Following the basis choice of [3], the Hamiltonian is:

$$H = -\sum_{\text{star}} Z\otimes Z\otimes Z\otimes Z \; - \sum_{\text{plaquette}} X\otimes X\otimes X\otimes X \quad (7)$$

where X and Z are Pauli matrices and the spin lattice is as shown in Fig.1(a). The spins live on the links (as this is a Z_2 lattice gauge theory); a "plaquette" refers to the four spins around a square while a "star" means the four spins around a vertex. We will work within the ground states of the star terms so we can effectively ignore them. Consider the spins $\{1, 2, 3, 4\}$ in Fig.1(a) and a θ-dependent Hamiltonian:

$$\begin{aligned} H_p(\theta) \;=\; & 1 + (\sin^2\theta - \cos^2\theta)(Z_1 + Z_2 + Z_3 + Z_4)/4 \\ & -2\sin\theta\cos\theta X_1X_2X_3X_4 \end{aligned} \quad (8)$$

We can check that within the subspace of $|0000\rangle$ and $|1111\rangle$ this has exactly the form of the driving term in (2) and that it takes $|0000\rangle$ to $\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ as θ goes from 0 to $\frac{\pi}{4}$. Other states in the Hilbert space, regardless of energy, do not affect the result because they are decoupled from the subspace, so the effective gap is 2 throughout the process. Since the ground state of the Hamiltonian (7) is known to be the superposition of all "closed strings" (plaquette excitations), the goal is to do the same operation of rotating the Z operators into $XXXX$ for every plaquette. Consider next the plaquette $\{4, 5, 6, 7\}$ after we have evolved $\{1, 2, 3, 4\}$. The Hamiltonian we use would have to be a little different since "4" is already entangled:

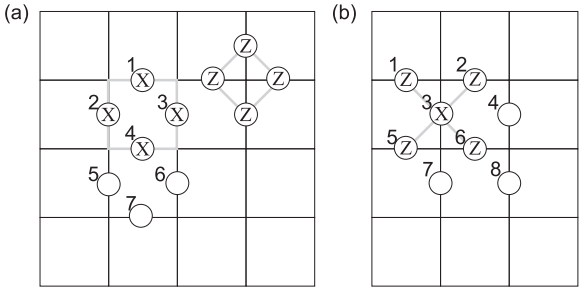

FIG. 1: (a) Interactions that appear in the toric code Hamiltonian. (b) Interactions in the cluster state Hamiltonian, following the same notation.

$$\begin{aligned} H'_p(\theta) \;=\; & 1 + (\sin^2\theta - \cos^2\theta)(Z_5 + Z_6 + Z_7)/3 \\ & -2\sin\theta\cos\theta X_4X_5X_6X_7 \end{aligned} \quad (9)$$

It is easy to verify that this Hamiltonian takes $\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)\,|000\rangle$ to $\frac{1}{2}(|0000000\rangle + |0001111\rangle + |1111000\rangle + |1110111\rangle)$ as θ goes from 0 to $\frac{\pi}{4}$. Generally, we can add a plaquette excitation to the superposition if at least one out of the four spins is unentangled. Bearing this in mind, we can perform these adiabatic rotations on plaquettes in parallel (coefficients in the Hamiltonian can be also adjusted to give a more translationally symmetrical appearance). For example, on a square lattice, we can operate on alternate rows of plaquettes in parallel. Dividing each row into odd and even plaquettes, this can be accomplished in two time steps. Then we operate on the remaining rows sequentially - to make sure we have an unentangled spin for every operation - which takes L steps if the lattice has $n = L \times L$ spins. The genus of the surface does not really affect this algorithm, except we have to notice that because the same loop can be viewed from two sides in a torus, there will be some plaquette operators that do not lead to new states in the superposition. As for other topologically inequivalent ground states, they can be generated in this manner by reversing the Z operators around an incontractible loop at the beginning. Compared to the linear interpolation, our evolution path is certainly more complicated. But it provides an interesting perspective, as it does not rely on duality mapping to an Ising model. In principle the technique we use can be generalized to prepare any string-net condensed state[8] because they are uniform superpositions of group elements generated by local string-nets.

The same trick also works for the cluster state[13]. The cluster state used in measurement-based quantum com-

puting, if put on a square lattice, is the superposition $\frac{1}{2^n}\sum_{z_i=\{0,1\}}(-1)^r |z_1...z_n\rangle$ where r is the number of adjacent "1"'s in the configuration $\{z_1...z_n\}$. It is stabilized by the operators $\sum_s X_s(\bigotimes_{s'} Z_{s'})$ where s' denotes sites adjacent to site s and the tensor product is over all four adjacent sites. Since each term takes half the states in the superposition to the other half, they play a role similar to the plaquette operators in the toric code. We can set the Hamiltonian for the sites $\{1,2,3,5,6\}$ in Fig.1(b) to be something similar to (8):

$$\begin{aligned} H_c(\theta) &= 1 + (\sin^2\theta - \cos^2\theta)Z_3 \\ &\quad -2\sin\theta\cos\theta X_3 Z_1 Z_2 Z_5 Z_6 \end{aligned} \tag{10}$$

and take θ from 0 to $\frac{\pi}{4}$. The same operation can be repeated on sites $\{3,4,6,7,8\}$, for instance. But since we always have an unentangled spin every time we turn on a stabilizer term, unlike the toric code case there is no need to operate sequentially. Therefore we can operate on all the odd sites in the first time step and all the even site in the second time step, completing the algorithm in $O(1)$ time.

Note that nothing prevents us from applying perturbation gadgets [14] on the stabilizers and turning our Hamiltonian into a nearest-neighbor one. Thus this is comparable with the linear interpolation result of [4], with the difference that we are preparing the cluster state rather than a state that can be used as one.

IV. THE HISTORY STATE

In all our examples so far the effective energy gap remains constant, so essentially we are implementing quantum circuits in the sense of [6]. Now let us consider an example where the gap shrinks as the system grows. In [5], it is shown that the history of a quantum circuit - a uniform superposition of intermediate states, appropriately made orthogonal - can be generated via an adiabatic linear interpolation. Up to small corrections unimportant for our purpose, the initial and final Hamiltonians are:

$$\begin{aligned} H_i &= \sum_{t=1}^{L} |\gamma_t\rangle\langle\gamma_t| \\ H_f &= \frac{1}{2}\sum_{t=0}^{L-1} |\gamma_t\rangle\langle\gamma_t| + |\gamma_{t+1}\rangle\langle\gamma_{t+1}| \\ &\quad - |\gamma_t\rangle\langle\gamma_{t+1}| - |\gamma_{t+1}\rangle\langle\gamma_t| \end{aligned} \tag{11}$$

where $|\gamma_t\rangle$'s represent the orthogonal intermediate states at time steps labeled by t. This adiabatic evolution takes the starting state $|\gamma_0\rangle$ to the history state $\frac{1}{\sqrt{L+1}}\sum_{t=0}^{L}|\gamma_t\rangle$. An obvious alternative to linear interpolation from H_i to H_f would be to apply adiabatic rotation step by step and rotate each term in H_i to a corresponding term in H_f:

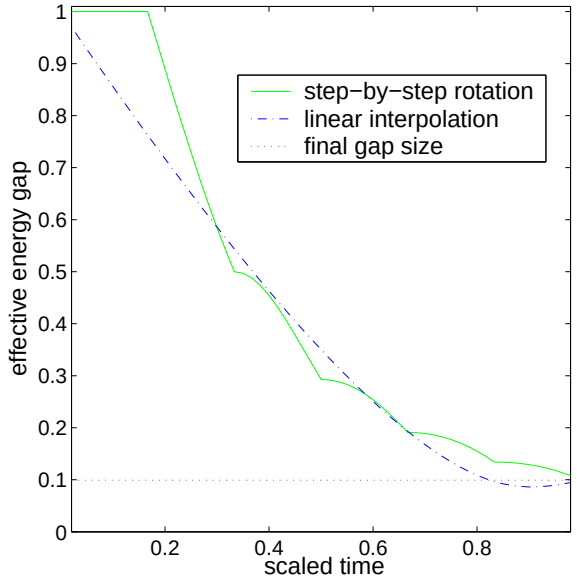

FIG. 2: Comparison of energy gap between stepwise adiabatic rotation to linear interpolation in the generation of history state with L=6. For the stepwise adiabatic rotation, the gap never crosses the line corresponding to the final gap. (Color online)

$$\begin{aligned} H_t(\theta_t) &= \sin^2\theta_t |\gamma_t\rangle\langle\gamma_t| + \cos^2\theta_t |\gamma_{t+1}\rangle\langle\gamma_{t+1}| \\ &\quad - \sin\theta_t\cos\theta_t(|\gamma_t\rangle\langle\gamma_{t+1}| + |\gamma_{t+1}\rangle\langle\gamma_t|) \end{aligned} \tag{12}$$

where as usual, θ_t goes from 0 to $\frac{\pi}{4}$ at each time step. Fig.2 shows the energy gap as a function of time for $L=6$ and compares this stepwise method to the linear interpolation. We can see that the minimal gap occurs at the very end. In [5] sophisticated techniques were used to estimate the scaling of the minimal gap as a function of L. It is much easier in our case, even compared to the simpler analysis of [11]. Observe that the final Hamiltonian is the discrete version of a kinetic energy term for a free particle in a box of size L. For large L, its spectrum simply consists of standing waves with energy proportional to the square of momentum, which goes as the inverse of wavelength. Thus we immediately conclude that the gap is $O(L^{-2})$. Now consider the speed of evolution as measured by $d\theta/dt$ for fixed error rate. Naively, it would scale as the square of the gap, i.e. $O(L^{-4})$. But we can use the bound in Eq.6 to get a better result. At the minimal gap, $g_1 \sim O(L^{-2})$ and $A \sim O(L^{-1})$. Since the excited states are standing waves, it is easy to see that $c_{mk}^2 \sim O(L^{-1})$. Altogether, we obtain $d\theta/dt \sim O(L^{-2})$.

Actually there is a more elegant way of using adiabatic rotation in this example. So far we have been taking θ from 0 to $\frac{\pi}{4}$. If we instead consider θ evolving from $\frac{\pi}{2}$ to $\frac{\pi}{4}$, we would take a single basis state to a uniform

superposition. This suggests that we try:

$$\begin{aligned} H(\theta) &= \cos^2\theta\,|\gamma_0\rangle\langle\gamma_0| + \sin^2\theta\,|\gamma_1\rangle\langle\gamma_1| \\ &\quad -\sin\theta\cos\theta(|\gamma_0\rangle\langle\gamma_1| + |\gamma_1\rangle\langle\gamma_0|) \\ &\quad +\frac{1}{2}\sum_{t=1}^{L-1} |\gamma_t\rangle\langle\gamma_t| + |\gamma_{t+1}\rangle\langle\gamma_{t+1}| \\ &\quad -|\gamma_t\rangle\langle\gamma_{t+1}| - |\gamma_{t+1}\rangle\langle\gamma_t| \end{aligned} \tag{13}$$

Here $|\gamma_0\rangle$ plays the role of $|\gamma_m\rangle$ in Eq.3; by rotating θ from $\frac{\pi}{2}$ to $\frac{\pi}{4}$, we achieve the desired evolution, remarkably, leaving most terms constant. The only thing we need to check is the energy gap at the beginning, where the Hamiltonian is a kinetic energy term plus a delta function potential of two energy units at one end. This is still easy to analyze compared to the linear interpolation. For sufficiently large L, the constant potential just amounts to a zero boundary condition for the low lying wavefunctions. The lowest energy state in the "box" is a standing wave with wavelength $4L$, so the gap is also $O(L^{-2})$. If we wish to implement the mirror evolution described in [6] to get from the history state to $|\gamma_L\rangle$, a similar rotation can be used on the other end of the "box".

V. QUANTUM SEARCH AND OPTIMIZATION

The preparation method above generates the ground state of a Hamiltonian; as such it is suggestive of a problem-solving algorithm. Indeed, let us consider an example that can be analyzed exactly. In [2] an adiabatic search algorithm was presented using linear interpolation to an oracle-like problem Hamiltonian. It turns out that one can formulate an alternative version using adiabatic rotation. The only extra resource we need is a non-computational state that the problem Hamiltonian annihilates (we may, for instance, add a non-computational state to each qubit to form a qutrit). Call this state $|i\rangle$ and our setup looks as follows:

$$\begin{aligned} H(\theta) &= (I - |m\rangle\langle m|) + \sin^2\theta\,|\psi_0\rangle\langle\psi_0| + \cos^2\theta\,|i\rangle\langle i| \\ &\quad -\sin\theta\cos\theta(|\psi_0\rangle\langle i| + |i\rangle\langle\psi_0|) \end{aligned} \tag{14}$$

where we have followed the notations of [2]: $|m\rangle$ is an unknown solution state we want to obtain and $|\psi_0\rangle$ is a known state that has non-zero overlap with $|m\rangle$. The identity operator "I" only acts on the computational space and is zero on $|i\rangle$. Denote the overlap by $a_0 \equiv \langle\psi_0|m\rangle$ (i.e. $a_0 = \frac{1}{\sqrt{N}}$ in [2]). Adiabatically changing θ from $\frac{\pi}{2}$ to 0 evolves the state $|i\rangle$ to $|m\rangle$. The energy gap $g_1(\theta) = 1 - \sin\theta\sqrt{1-a_0^2}$, which at $\theta = \frac{\pi}{2}$ is $O(a_0^2) \sim O(N^{-1})$ if $a_0 = \frac{1}{\sqrt{N}}$. This appears different from the case in [2] where the minimal gap is $O(N^{-\frac{1}{2}})$. We must however take into account $\left|\langle k|\frac{dH}{d\theta}|0\rangle\right|$. The first excited state is proportional to $a_0|m\rangle + (\sin\theta\sqrt{1-a_0^2} - 1)|\psi_0\rangle - \cos\theta\sqrt{1-a_0^2}|i\rangle$; explicit calculation yields:

$$\begin{aligned} \left|\langle 1|\frac{dH}{d\theta}|0\rangle\right| &\le a_0\frac{\left|\sin\theta - \sqrt{1-a_0^2}\right|}{1-\sin\theta\sqrt{1-a_0^2}} \\ &\sim O(a_0). \end{aligned} \tag{15}$$

Now use Eq.4 to set the rotation speed: $d\theta \sim (g_1^2/a_0)dt$. Integrating this from $\theta = \frac{\pi}{2}$ to $\theta = 0$ and setting $a_0 = \frac{1}{\sqrt{N}}$, we obtain the total evolution time $T \sim (\sqrt{N} - \sqrt{N-1})^{-1} \sim O(\sqrt{N})$ for large N. Quadratic speedup is achieved as promised.

In principle we may replace $I - |m\rangle\langle m|$ in Eq.14 by other problem Hamiltonians to find other ground states. This would work as long as the gap at $\frac{\pi}{2}$ is sufficiently large. But if we want to solve NP-complete problems as in [1], notice that $|\psi_0\rangle\langle\psi_0|$ is generally nonlocal and does not respect the bit structure [15]. Suppose we set $|i\rangle = |222...\rangle$ by using qutrits with an extra state $|2\rangle$ and assume $|m\rangle = |000...\rangle$ without loss of generality, it can be proved that no local Hamiltonian can have a unique ground state of the form

$$\frac{1}{\sqrt{1 + a_0^2\tan^2\theta}}(|000...\rangle + a_0\tan\theta\,|222...\rangle)$$

by arguments in [6]. Since locality is important to any realistic algorithm, we need a local version of adiabatic rotation different from (1)-(3). For example, if $|\psi_0\rangle = \bigotimes_n |+\rangle$ where n is the number of qubits and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, we may want to replace the driving term by $\sum_n U|+\rangle\langle+|U^\dagger$ where U rotates between $|+\rangle$ and $|2\rangle$. We may then expect the ground state to contain a piece proportional to $\bigotimes_n(|+\rangle + \tan\theta\,|2\rangle)$. The appearance of an extra $(3^n - 2^n)$ states, however, can significantly change the spectrum. More work would be required to find a feasible local model.

VI. DISCUSSION

The issue of locality arises in the previous section because our rotation method requires the initial state to be separated from the computational states. The same requirement applies to the history state preparation in Section IV, but there locality was preserved by use of clock qubits (which ensures the orthgonality of the $|\gamma_t\rangle$s' - we refer the readers to [5] for details). This suggests that clever use of ancilla would be important to the design of a local adiabatic rotation algorithm. We should note that the form of the driving term (2) is largely chosen for simplicity and can be generalized. The underlying idea is that when we have an efficient way of adiabatically evolving from state A to state B, we may combine it with a time-independent "diffusion" Hamiltonian H_0 to

attain a more complicated state B' - a state that overlaps B but we otherwise may not know how to get to. Such a perspective can be useful for algorithm design.

Results in this article also raise another issue. In almost all our examples, adiabatic rotations turn out to have the same efficiency as the linear interpolations - not only up to the same complexity class, but down to the order of polynomial. One might wonder if there is some deep connection between different adiabatic paths under the same set of Hamiltonian interactions. If such connection exists, might it be possible to prove the efficiency of a linear interpolation path by proving the efficiency of a long-winded path that is easier to analyze? At this point this is an entirely open question.

[1] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, arXiv:quant-ph/0001106

[2] J. Roland and N. Cerf, Phys. Rev. A 65, 042308 (2002); See also W. van Dam, M. Mosca, U. Vazirani, Proc. 42nd Symposium on Foundations of Computer Science, 279 (2001)

[3] A. Hamma and D. Lidar, arXiv:quant-ph/0607145

[4] S. D. Bartlett, T. Rudolph, Phys. Rev. A, 74, 040302(R) (2006)

[5] D. Aharonov, W. Van Dam, J. Kempe, Z. Landau, and S. Lloyd, O. Regev, arXiv:quant-ph/0405098

[6] M. S. Siu, Phys. Rev. A 71, 062314 (2005)

[7] A. Kitaev, Annals Phys. 303, 2 (2003)

[8] M. Levin and X.-G. Wen, Phys. Rev. B. 71, 045110 (2005)

[9] E. Dennis, A. Kitaev, A. Landahl and J. Preskill, J. Math. Phys. 43, 4452 (2002)

[10] S. Bravyi, M. Hastings, F. Verstraete, Phys. Rev. Lett. 97, 050401 (2006)

[11] P. Deift, M.-B. Ruskai and W. Spitzer, arXiv:quant-ph/0605156

[12] A. Messiah. Quantum Mechanics. John Wiley & Sons, New York, 1958; A more rigorous analysis can be found in S. Jansen, M.-B. Ruskai, R. Seiler, arXiv:quant-ph/0603175

[13] R. Raussendorf and H. Briegel. Phys. Rev. Lett. 86, 5188 (2001); M. Nielsen, arXiv:quant-ph/0504097

[14] J. Kempe, A. Kitaev and O. Regev, arXiv:quant-ph/0406180 ; R. Oliveira and B. Terhal, arXiv:quant-ph/0504050

[15] E. Farhi, J. Goldstone, S. Gutmann and D. Nagaj, arXiv:quant-ph/0512159